TRINITY COLLEGE

Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed 65% (97 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

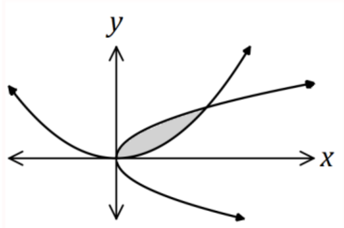
Working time: 100 minutes.

Question 9 (5 marks)

The region R enclosed by the curves $y^2 = ax$ and $x^2 = 8ay$, has an area of 1014 square units.

Determine the value of the positive constant a.

Solution



Intersect at $(0,0)$ and $(4a, 2a)$ (CAS)

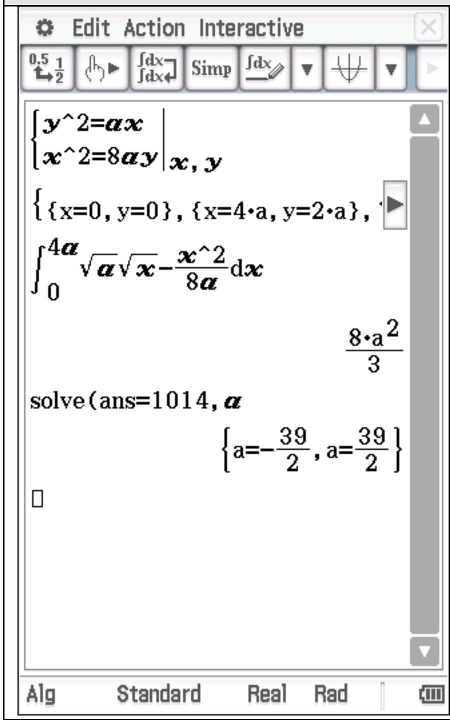
$$A = \int_0^{4a} (\sqrt{a}\sqrt{x}) - \left(\frac{x^2}{8a}\right) dx$$
$$= \frac{8a^2}{3}$$

$$\frac{8a^2}{3} = 1014 \Rightarrow a = \frac{39}{2} = 19.5$$

Specific behaviours

✓ sketches curves
✓ identifies points of intersection
✓ correctly formed integral
✓ evaluates integral in terms of a
✓ solves for a

CAS solution



Question 10 **(4 marks)**

A sphere has diameter AB where points A and B have position vectors $(2, 0, 3)$ and $(0, 8, 9)$ respectively.

(a) Determine the vector equation of the sphere. (2 marks)

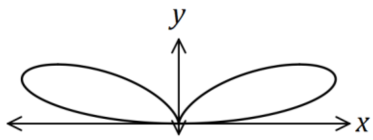
Solution
Centre $\begin{pmatrix}1\\4\\6\end{pmatrix}$ and radius $\left\|\begin{pmatrix}-1\\4\\3\end{pmatrix}\right\| = \sqrt{26}$ $\left\|\mathbf{r} - \begin{pmatrix}1\\4\\6\end{pmatrix}\right\| = \sqrt{26}$ $(\sqrt{26} \approx 5.1)$
Specific behaviours
✓ determines radius   ✓ correct centre and vector equation

(b) State, with justification, whether the point P with position vector $(-1, 1, 2)$ lies inside, outside or on the surface of the sphere. (2 marks)

Solution
$\left\|\begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}1\\4\\6\end{pmatrix}\right\| = \sqrt{4 + 9 + 16} = \sqrt{29}$ $\sqrt{29} > \sqrt{26} \Rightarrow P$ is outside sphere $(\sqrt{29} \approx 5.4)$
Specific behaviours
✓ determines distance from centre   ✓ correct conclusion

Question 11 **(7 marks)**

(a) A bifolium has equation $(x^2 + 2y^2)^2 = 9x^2y$.



Show that the gradient of the bifolium at the point $(2, 1)$ is -1. (4 marks)

Solution
$2(x^2 + 2y^2)(2x + 4yy') = 18xy + 9x^2y'$ $x = 2, y = 1 \Rightarrow 2(4 + 2)(4 + 4y') = 36 + 36y'$ $48 + 48y' = 32 + 36y'$ $12y' = -12$ $y' = -1$
Specific behaviours
✓ implicit diff of RHS ✓ implicit diff of LHS ✓ substitutes ✓ simplifies

(b) The gradient of a circle that passes through the point $(2, 4)$ is given by

$$\frac{dy}{dx} = \frac{2}{y} - \frac{x}{y}.$$

Determine the equation of the circle. (3 marks)

Solution
$\int y\ dy = \int (2 - x)\, dx$ $\frac{y^2}{2} = 2x - \frac{x^2}{2} + k$ $y^2 = 4x - x^2 + c$ $c = 4^2 - 8 + 4 = 12$ $x^2 - 4x + y^2 = 12$
Specific behaviours
✓ separates variables ✓ integrates ✓ correct equation, no specific form required

Question 12 **(8 marks)**

(a) Bags of oranges are packaged for sale by a supermarket. The population mean and standard deviation of the weight of the bags is known to be 3.15 kg and 25 g respectively.

Determine the probability that the total weight of a random sample of 40 bags of oranges is no greater than 125.7 kg. (4 marks)

Solution
Let $\overline{W}$ be the distribution of random samples of size 40 from the population. Then $\overline{W} \sim N\left(3.15, \frac{0.025^2}{40}\right) \sim N(3.15, 0.00395^2)$ $P\left(\overline{W} \leq \dfrac{125.7}{40}\right) = P(\overline{W} \leq 3.1425) = 0.0289$
Specific behaviours
✓ defines sample mean as a normally distributed rv ✓ indicates parameters of normal distribution ✓ indicates probability calculated ✓ correct probability

(b) The supermarket also packs bags of lemons for sale. The weights of the bags have a population mean and standard deviation of μ and σ kg respectively.

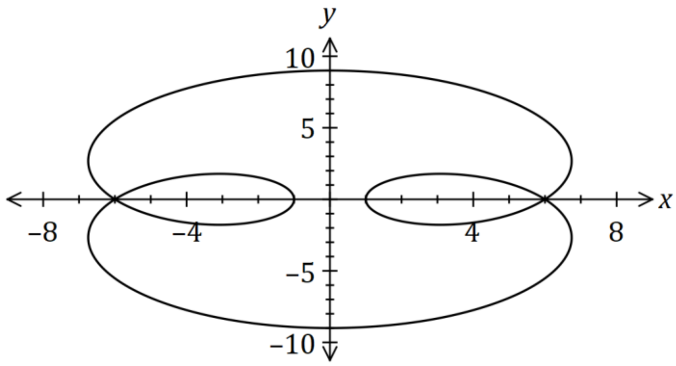
A random sample of 48 bags was taken and used to construct a 90% confidence interval for μ. If the interval was $(1.996, 2.034)$, determine an estimate for σ. (4 marks)

Solution
Margin of error: $\frac{2.034 - 1.996}{2} = 0.019$ $90\% \Rightarrow z = 1.645$ $\dfrac{\sigma}{\sqrt{48}} \times 1.645 = 0.019$ $\sigma = 0.080 \text{ kg}$
Specific behaviours
✓ calculates margin of error ✓ uses correct z-score ✓ writes equation for margin of error ✓ correct standard deviation

Question 13 **(7 marks)**

The position vector $\mathbf{r}$ at time t seconds of a small particle P is shown below and given by

$$\mathbf{r}(t) = (5\cos(t) - 4\cos(3t))\mathbf{i} + (5\sin(t) - 4\sin(3t))\,\mathbf{j}\text{ cm.}$$



(a) Determine the change in displacement of P between $t = 0$ and $t = \frac{\pi}{2}$. (2 marks)

Solution
$\mathbf{r}(0) = \mathbf{i}, \quad \mathbf{r}\left(\frac{\pi}{2}\right) = 9\mathbf{j}, \quad \Delta\mathbf{r} = -\mathbf{i} + 9\mathbf{j}$ cm
Specific behaviours
✓ determines positions ✓ states change

(b) Determine the velocity vector of P when $t = \frac{\pi}{2}$. (2 marks)

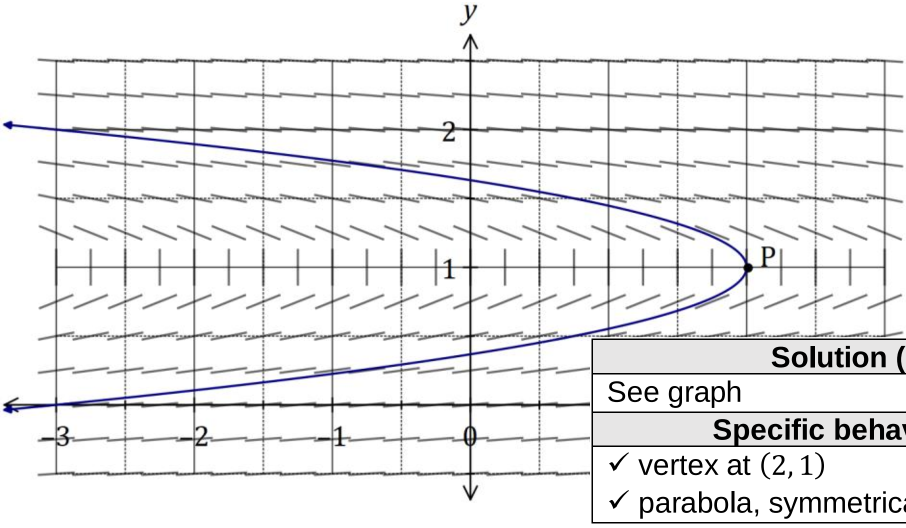
Solution
$\mathbf{v}(t) = (-5\sin(t) + 12\sin(3t))\,\mathbf{i} + (5\cos(t) - 12\cos(3t))\mathbf{j}$ $\mathbf{v}\left(\frac{\pi}{2}\right) = -17\mathbf{i}$ cm/s
Specific behaviours
✓ differentiates to obtain velocity vector ✓ states velocity vector

(c) Determine the total distance travelled by P until it first returns to its initial position. (3 marks)

Solution
Period is 2π $d = \int_0^{2\pi} \lvert\mathbf{v}(t)\rvert\,dt$ $= 78.71$ cm
Specific behaviours
✓ determines time to return ✓ correct integral ✓ distance (that rounds to 79 cm).

Question 14 **(8 marks)**

The slope field for the differential equation $\frac{dy}{dx} = \frac{-1}{10(y-1)}$ is shown below.



Solution (a)
See graph
Specific behaviours
✓ vertex at $(2, 1)$
✓ parabola, symmetrical about $y = 1$

(a) Sketch the solution of the differential equation that passes through the point $P(2, 1)$. (2 marks)

A different solution of the differential equation passes through the points $A(4.2, -1)$ and $B(4.3, b)$.

(b) Use the increments formula to estimate the value of b. (3 marks)

Solution
$\delta x = 0.1 = \frac{1}{10}, \quad \frac{dy}{dx} = \frac{-1}{10(-1-1)} = \frac{1}{20}$ $\delta y \approx \frac{1}{20} \times \frac{1}{10} \approx \frac{1}{200} = 0.005$ $b \approx -1 + 0.005 \approx -0.995$
Specific behaviours
✓ calculates gradient at A ✓ calculates δy using increments formula   ✓ correct estimate

(c) Calculate the value of the second derivative of the solution through A. (3 marks)

Solution
$\frac{dy}{dx} = \frac{-1}{10}(y-1)^{-1}$ $\frac{d^2y}{dx^2} = \frac{1}{10}(y-1)^{-2} \times \frac{dy}{dx}$ $= \frac{1}{40} \times \frac{1}{20} = \frac{1}{800}$
Specific behaviours
✓✓ expression for second derivative   ✓ correct value

Question 15 **(8 marks)**

Using the given substitution, rewrite the following integrals in terms of u and then evaluate.

(a) $\int_0^{\pi} \sin^3\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$, using $u = \sin\left(\frac{x}{2}\right)$. (4 marks)

Solution
$du = \frac{1}{2}\cos\left(\frac{x}{2}\right)dx$ $x = 0, u = 0, \quad x = \pi, u = 1$ $I = \int_0^{\pi} 2\sin^3\left(\frac{x}{2}\right) \times \frac{1}{2}\cos\left(\frac{x}{2}\right)dx = \int_0^1 2u^3\,du$ $I = \left[\frac{u^4}{2}\right]_0^1 = \frac{1}{2}$
Specific behaviours
✓ relates du and dx ✓ replaces bounds of integration   ✓ expresses integrand in terms of u ✓ evaluates

(b) $\int_1^7 \frac{2x}{\sqrt{2x+2}}dx$, using $u = \sqrt{2x+2}$. (4 marks)

Solution
$u^2 = 2x + 2 \Rightarrow 2u\,du = 2\,dx \Rightarrow u\,du = dx$ $x = 1, u = 2; \;\; x = 7, u = 4$ $I = \int_1^7 \frac{2x}{\sqrt{2x+2}}\,dx$ $= \int_2^4 \frac{u^2-2}{u}\,u\,du$ $= \int_2^4 u^2 - 2\;\,du$ $= \left[\frac{u^3}{3} - 2u\right]_2^4 = \frac{44}{3}$
Specific behaviours
✓ relates du and dx ✓ replaces bounds of integration   ✓ simplifies integrand in terms of u ✓ evaluates

Question 16 **(9 marks)**

The durations, in minutes, of a sample of 10 calls to an IT support line were as follows.

$$29, \quad 14, \quad 11, \quad 18, \quad 30, \quad 9, \quad 22, \quad 37, \quad 24, \quad 19.$$

The duration of calls to the support line has a known standard deviation of 6 minutes 50 seconds.

(a) Stating two necessary assumptions, construct a 95% confidence interval for the mean duration of calls to the support line. (7 marks)

Solution
(i) **Sample** is **random** (ii) **Durations** are **normal**(ly distributed) $\bar{x} = \dfrac{213}{10} = 21.3$ $95\% \Rightarrow z = 1.96$ $se = \dfrac{6.8\overline{3}}{\sqrt{10}} \approx 2.161$ $21.3 \pm 1.96 \times \dfrac{6.8\overline{3}}{\sqrt{10}} \Rightarrow 21.3 \pm 4.2353$ $(17.0 \text{ to } 17.1, 25.5 \text{ to } 25.6)$
Specific behaviours
✓ assumption (i) (must use both bolded words) ✓ assumption (ii) (must use both bolded words) ✓ calculates sample mean ✓ indicates correct z-score ✓ indicates correct standard error ✓ indicates interval construct ✓ calculates interval within ranges shown

(b) Comment, with justification, on a claim that the mean duration of calls to the support line is 18 minutes. (2 marks)

Solution
18 minutes lies within the CI and so claim is reasonable.
Specific behaviours
✓ refers to 18 relative to CI ✓ comment supported by reference

Question 17 **(7 marks)**

A company recently introduced a new electronic control device for homes. In one city, the number of households H, in thousands, that own the device t months after observations began can be modelled by

$$H(t) = \frac{30}{1 + 2e^{-0.05t}}, \qquad t \geq 0.$$

(a) Use the model to determine

(i) the maximum number of households expected to own the device. (1 mark)

Solution
$H(\infty) = 30 \Rightarrow 30\ 000$ households
Specific behaviours
✓ correct number

(ii) how long it will take for the number of households owning the device to double from the initial number. (2 marks)

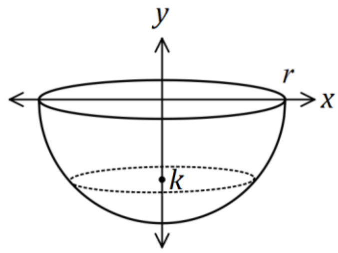
Solution
$H(0) = 10 \Rightarrow 20 = \dfrac{30}{1 + 2e^{-0.05t}}$ $t = 27.7$ months
Specific behaviours
✓ initial number ✓ correct time

(b) Show that the rate of change of the population satisfies the equation $H'(t) = kH(30 - H)$ and determine the value of the constant k. (4 marks)

Solution
$H = 30(1 + 2e^{-0.05t})^{-1}$ $H'(t) = -30(1 + 2e^{-0.05t})^{-2}(2e^{-0.05t})(-0.05)$ But $1 + 2e^{-0.05t} = \dfrac{30}{H}$ $H'(t) = -30\left(\dfrac{30}{H}\right)^{-2}\left(\dfrac{30}{H} - 1\right)(-0.05)$ $= \dfrac{0.05H^2}{30}\left(\dfrac{30}{H} - 1\right)$ $= \dfrac{H}{600}(30 - H)$ $k = \dfrac{1}{600}\ (\approx 0.001\bar{6})$
Specific behaviours
✓ correct derivative of H ✓ substitutes for denominator of H ✓ systematic simplification ✓ value of k

Question 18 **(10 marks)**

The inner surface of a hemispherical bowl can be modelled by rotating part of the circle with equation $x^2 + y^2 = r^2,\ y \le 0$, about the y axis.



With the circular rim level, a liquid is poured into the hemisphere to a depth of k, measured from the bottom of the hemisphere, where $0 \le k \le r$.

(a) Write a definite integral in terms of r, k and y for the volume of liquid in the bowl. (2 marks)

Solution
$V = \int_{-r}^{-r+k} \pi(r^2 - y^2)\, dy \quad \left(= \int_{r-k}^{r} \pi(r^2 - y^2)\, dy\right)$
Specific behaviours
✓ correct integrand and dy ✓ correct limits

(b) Use your answer to (a) to show that the volume of liquid in a bowl when it is filled to a depth k is given by $\frac{1}{3}\pi k^2(3r - k)$. (3 marks)

Solution
$V = \pi\left[r^2 y - \frac{y^3}{3}\right]_{-r}^{-r+k}$ $= \frac{\pi}{3}[(3r^2(k-r) - (k-r)^3) - (3r^2(-r) - (-r)^3)]$ $= \frac{\pi}{3}[3r^2k - 3r^3 - (k^3 - 3k^2r + 3kr^2 - r^3) - (-3r^3 + r^3)]$ $= \frac{\pi}{3}[3r^2k - 3r^3 - hk^3 + 3k^2r - 3kr^2 + r^3 + 2r^3]$ $= \frac{\pi}{3}[-k^3 + 3k^2r]$ $= \frac{1}{3}\pi k^2(3r - k)$
Specific behaviours
✓ correct antiderivative and substitution of limits seen ✓ correct expansion of $(k-r)^3$ seen (or $(r-k)^3$) ✓ correct simplification seen

(c) A hemispherical bowl, with an internal radius of 36 cm, is filled with water at a constant rate from empty to full in 432 seconds. Determine the rate of increase of the depth of water at the instant the hemisphere contains $4\,608\pi$ cm^3 of water. (5 marks)

Solution
$\frac{dV}{dt}=\frac{2}{3}\pi(36)^3\div 432=72\pi\ \text{cm}^3/\text{s}\ (\approx 226.2)$ $V=4608\pi=\frac{1}{3}\pi k^2(3(36)-k)\Rightarrow k=12$ $\frac{dV}{dk}=\pi(2rk-k^2)$ $=\pi(2(36)(12)-12^2)$ $=720\pi\ (\approx 2262)$ $\frac{dk}{dt}=\frac{dk}{dV}\times\frac{dV}{dt}$ $=\frac{1}{720\pi}\times 72\pi$ $\frac{dk}{dt}=\frac{1}{10}\ \text{cm/s}\ (=0.1)$
Specific behaviours
✓ calculates dV/dt ✓ calculates height   ✓ calculates dV/dk ✓ uses chain rule   ✓ correct rate

Question 19 **(11 marks)**

Small bodies P and Q are initially at $A(6, 3, -7)$ and $C(4, -3, -1)$ respectively and are travelling with constant velocities.

One second later, P and Q are at $B(4, 1, -2)$ and $D(3, -2, 1)$ respectively.

(a) Determine the vector equation for the path of P at any time t, where $t = 0$ when P is at A. (2 marks)

Solution
$\overrightarrow{AB} = (4, 1, -2) - (6, 3, -7) = (-2, -2, 5)$ $\mathbf{r}_P = (6, 3, -7) + t(-2, -2, 5)$
Specific behaviours
✓ direction vector   ✓ correct equation

(b) Show that the paths of P and Q cross, stating the point of intersection and explaining whether they also collide. (6 marks)

Solution
$\overrightarrow{CD} = (3, -2, 1) - (4, -3, -1) = (-1, 1, 2)$ $\mathbf{r}_Q = (4, -3, -1) + s(-1, 1, 2)$ $\mathbf{i}$ coeffs: $6 - 2t = 4 - s$ $\mathbf{j}$ coeffs: $3 - 2t = -3 + s$ $\therefore t = 2, \quad s = 2$ $\mathbf{r}_P(2) = (6, 3, -7) + 2(-2, -2, 5) = (2, -1, 3)$ $\mathbf{r}_Q(2) = (4, -3, -1) + 2(-1, 1, 2) = (2, -1, 3)$ Since $\mathbf{k}$ coefficients are both 3, then paths cross.   Also, P and Q collide as they are at intersection at the same time.
Specific behaviours
✓ equation for path of Q ✓ equates $\mathbf{i}$ and $\mathbf{j}$ coefficients   ✓ solves for times   ✓ checks $\mathbf{k}$ coefficents for consistency   ✓ states point of intersection   ✓ states that paths cross, explains why collide

(c) A third small body G is stationary at the point $(-2, -3, 12)$. Determine whether G lies in the same plane as the paths of P and Q. (3 marks)

Solution
$$-\mathbf{n} = \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ -1 \\ -4 \end{pmatrix}$$ $$\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 1 \\ 4 \end{pmatrix} = 29$$ $$\begin{pmatrix} -2 \\ -3 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 1 \\ 4 \end{pmatrix} = 48 - 18 - 3 = 27 \Rightarrow \text{Not in same plane}$$
Specific behaviours
✓ determines normal to plane ✓ determines equation of plane ✓ substitutes point and draws conclusion

Question 20 **(6 marks)**

(a) Determine the cube roots of $4\sqrt{2}(1-i)$, giving your solutions in polar form $r\,\text{cis}\,\theta$ where $-\pi < \theta \le \pi$.

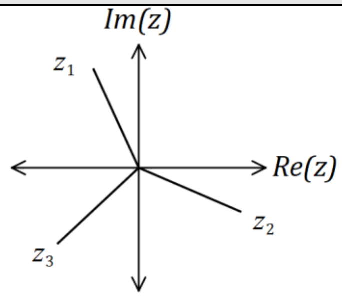
(3 marks)

Solution
$z^3 = 4\sqrt{2}(1-i)$ $= 8\,\text{cis}\left(-\frac{\pi}{4}\right)$ $z = 2\,\text{cis}\left(\frac{2n\pi}{3} - \frac{\pi}{12}\right), \quad n = -1, 0, 1$ $z_1 = 2\,\text{cis}\left(\frac{7\pi}{12}\right), \quad z_2 = 2\,\text{cis}\left(-\frac{\pi}{12}\right), \quad z_3 = 2\,\text{cis}\left(-\frac{3\pi}{4}\right)$ $(Arg(z) = -135°, -15°, 105°)$
Specific behaviours
✓ expresses in polar form   ✓ one correct root   ✓ all 3 roots

(b) One of the cube roots of $4\sqrt{2}(1-i)$ is also a fourth root of w.

If ϕ is the argument of a fourth root of w that lies in the first quadrant $\left(0 \le \phi \le \frac{\pi}{2}\right)$, determine all possible values of ϕ. (3 marks)

Solution



w has four roots evenly spaced at $\frac{\pi}{2}$, one of which is either $z_1, z_2,$ or z_3.

$$z_1 - \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{12}, \quad z_2 + \frac{\pi}{2} \Rightarrow \phi = \frac{5\pi}{12}, \quad z_3 + \pi \Rightarrow \phi = \frac{\pi}{4}$$

$(\phi = 15°, 45°, 75°)$

Specific behaviours

✓ sketch of cube roots
✓ one correct value
✓ all possible values

Question 21 **(7 marks)**

A particle moves with velocity v in a straight line so that its acceleration a is given by

$$a = 6v - 0.6v^2, \qquad v > 0.$$

Distances are measured in metres and times are in seconds. Initially the particle is at the origin ($x = 0$) and has velocity $v = 50$.

(a) Use $a = v\dfrac{dv}{dx}$ to express the velocity v of the particle as a function of its displacement x.

(6 marks)

Solution
$v\dfrac{dv}{dx} = v(6 - 0.6v)$ $\displaystyle\int \frac{-0.6}{6 - 0.6v}\,dv = \int -0.6\,dt$ $\ln\lvert 6 - 0.6v\rvert = -0.6x + c$ $0.6v - 6 = ae^{-0.6x}$ $x = 0, v = 50 \Rightarrow 0.6(50) - 6 = a \Rightarrow a = 24$ $\therefore v = 40e^{-0.6x} + 10$
Specific behaviours
✓ uses required form of acceleration   ✓ separates variables   ✓ integrates   ✓ writes in exponential form (*attn to removal of absolute value*)   ✓ determines constant   ✓ correct equation

(b) Determine the exact distance of the particle from the origin when its velocity $v = 15$.

(1 mark)

Solution
$40e^{-0.4x} + 10 = 15$ $x = 5\ln 2 \approx 3.466 \text{ m}$
Specific behaviours
✓ correct distance

End of questions

Supplementary page

Question number: __________

Supplementary page

Question number: _________